The spt-function of Andrews

William Y.C. Chen

Center for Applied Mathematics

Tianjin University, Tianjin 300072, P. R. China

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Goals for this talk

My goals for this talk include the following:

(1) The introduction to spt-functions;

(2) The spt-crank;

(3) More spt-congruences on spt-functions;

(4) Generalizations and variations on spt-functions;

(5) Asymptotic properties on spt-functions;

(6) Conjecture on inequalities on spt-functions.

The definition of the spt-function

The spt-function was introduced by Andrews as the weighted counting of partitions with respect to the number of occurrences of the smallest part. To be specific, for a partition λ, let $n_s(\lambda)$ denote the number of appearances of the smallest of λ, then

$$\mathrm{spt}(n) = \sum_{|\lambda|=n} n_s(\lambda).$$

Example

For $n = 4$, we see that $spt(4) = 10$. The five partitions of 4 and the values of $n_s(\lambda)$ are listed below:

λ	(4)	$(3,1)$	$(2,2)$	$(2,1,1)$	$(1,1,1,1)$
$n_s(\lambda)$	1	1	2	2	4

Dyson's rank

Recall that the rank of a partition was introduced by Dyson as given below.

Definition (Dyson, 1944)

Let λ be a partition. The rank of λ is defined to be the largest part of λ minus the number of parts of λ.

For example, the rank of $(5, 4, 2, 1, 1, 1)$ is equal to $5 - 6 = -1$.

F. Dyson, Some guesses in the theory of partitions, Eureka (Cambridge) 8 (1944) 10–15.

The definition of crank

In 1988, Andrews and Garvan gave the definition of crank for an ordinary partition as follows:

Definition (Andrews, Garvan, 1988)

For a partition λ, the crank of λ is defined as follows:

$$crank(\lambda) = \begin{cases} \lambda_1, & \text{if } n_1(\lambda) = 0; \\ \mu(\lambda) - n_1(\lambda), & \text{if } n_1(\lambda) > 0, \end{cases}$$

where $n_1(\lambda)$ denotes the number of 1's in λ and $\mu(\lambda)$ denotes the number of parts of λ larger than $n_1(\lambda)$.

Example

For example, let $\lambda = (7, 7, 6, 4, 3, 1, 1, 1, 1)$, then $n_1(\lambda) = 4$ and $\mu(\lambda) = 3$. This implies $crank(\lambda) = 3 - 4 = -1$.

The relation between spt-function and rank, crank

Andrews proved that

Theorem (Andrews, 2008)

For $n \geq 1$,

$$\mathrm{spt}(n) = \sum_{m=-\infty}^{\infty} m^2 M(m,n) - \sum_{m=-\infty}^{\infty} m^2 N(m,n),$$

where $M(m,n)$ denote the number of partitions of n with crank m, and $N(m,n)$ denote the number of partitions of n with rank m.

G.E. Andrews, The number of smallest parts in the partitions of n, J. Reine Angew. Math. 624 (2008) 133–142.

George E. Andrews and F.G. Garvan, Dyson's crank of a partition, Bull. Amer. Math. Soc. 18 (1988), 167–171.

The congruences on the spt-function

Andrews also deduced the following congruences on the spt-function

$$\begin{aligned}
\operatorname{spt}(5n+4) &\equiv 0 \pmod{5};\\
\operatorname{spt}(7n+5) &\equiv 0 \pmod{7};\\
\operatorname{spt}(13n+6) &\equiv 0 \pmod{13},
\end{aligned}$$

which is a striking resemblance to the Ramanujan's congruences.

S-partitions

To give a combinatorial interpretation of these three spt-congruences, Andrews, Garvan and Liang introduced the spt-crank which is defined on S-partitions.

Definition (Andrews, Garvan and Liang, 2012)

Let $\mathcal{D}$ denote the set of partitions into distinct parts and $\mathcal{P}$ denote the set of partitions. For $\pi \in \mathcal{P}$, we use $s(\pi)$ to denote the smallest part of π with the convention that $s(\emptyset) = +\infty$. Define

$$S = \{(\pi_1, \pi_2, \pi_3) \in \mathcal{D} \times \mathcal{P} \times \mathcal{P} \colon \pi_1 \neq \emptyset \text{ and } s(\pi_1) \leq \min\{s(\pi_2), s(\pi_3)\}\}.$$

The triplet $\pi = (\pi_1, \pi_2, \pi_3) \in S$ is called to be an S-partitions of n with weight $\omega(\pi) = (-1)^{\ell(\pi_1)-1}$ if $|\pi| = |\pi_1| + |\pi_2| + |\pi_3| = n$.

Example

The partition triplet $((3, 2), (4, 4, 3, 2), (3, 3, 3))$ is an S-partition of 27.

spt-crank

The spt-crank is defined as follows.

Definition (Andrews, Garvan and Liang, 2012)

Let π be an S-partition, the spt-crank of π, denoted $r(\pi)$, is defined to be the difference between the number of parts of π_2 and π_3, that is,

$$r(\pi) = \ell(\pi_2) - \ell(\pi_3).$$

Example

The S-partition $\pi = ((3,2),(4,4,3,2),(3,3,3))$ is an S-partition of 27 with weight $\omega(\pi) = -1$ and spt-crank $4 - 3 = 1$.

G.E. Andrews, F.G. Garvan and J.L. Liang, Combinatorial interpretations of congruences for the spt-function, Ramanujan J. 29 (2012) 321–338.

Notation

Let $N_S(m,n)$ denote the net number of S-partitions of n with spt-crank m, that is,

$$N_S(m,n) = \sum_{\substack{|\pi|=n \\ r(\pi)=m}} \omega(\pi)$$

and

$$N_S(k,t,n) = \sum_{m \equiv k \pmod{t}} N_S(m,n).$$

Combinatorial interpretations

Andrews, Garvan and Liang showed the following relations:

Theorem (Andrews, Garvan and Liang, 2013)

$$N_S(k,5,5n+4) = \frac{\mathrm{spt}(5n+4)}{5}, \quad \textit{for} \quad 0 \le k \le 4,$$
$$N_S(k,7,7n+5) = \frac{\mathrm{spt}(7n+5)}{7}, \quad \textit{for} \quad 0 \le k \le 6,$$

which imply $spt(5n+4) \equiv 0 \pmod 5$ and $spt(7n+5) \equiv 0 \pmod 7$.

Generating function for $N_S(m, n)$

Andrews, Garvan and Liang also showed that

Theorem (Andrews, Garvan and Liang, 2013)

$$\sum_{m=-\infty}^{+\infty}\sum_{n\geq 0} N_S(m,n)z^m q^n = 1+\sum_{m=0}^{\infty} z^m \sum_{j=0}^{\infty} \frac{q^{j^2+mj+2j+m+1}}{(q;q)_{j+m}}\times \sum_{h=0}^{j} \begin{bmatrix} j \\ h \end{bmatrix} \frac{q^{h^2+h}}{(q;q)_h(1-q^{m+1+j+h})}$$

$$+\sum_{m=1}^{\infty} z^{-m} \sum_{j=m}^{\infty} \frac{q^{j^2-mj+2j-m+1}}{(q;q)_{j-m}}\times \sum_{h=0}^{j} \begin{bmatrix} j \\ h \end{bmatrix} \frac{q^{h^2+h}}{(q;q)_h(1-q^{j-m+1+h})}.$$

Nonnegativity of $N_S(m,n)$

From the generating function of $N_S(m,n)$, Andrews, Garvan and Liang derived the nonnegativity of $N_S(m,n)$.

Theorem (Andrews, Garvan and Liang, 2013)

For all $n \geq 0$,

$$N_S(m,n) \geq 0.$$

Marked partitions

Andrews, Dyson and Rhoades introduced marked partitions.

Definition (Andrews, Dyson and Rhoades, 2013)

A marked partition of n is a pair (λ, k), where λ is an ordinary partition of n and k is an integer identifying one of its minimum parts.

Here we set $\lambda_k = s(\lambda)$ where $s(\lambda)$ denotes the minimum part of λ.

G.E. Andrews, F.J. Dyson and R.C. Rhoades, On the distribution of the spt-crank, Mathematics 1 (2013) 76–88.

Example

For example, there are ten marked partitions of 4.

	$\lambda \in \mathcal{P}(4)$	$n_s(\lambda)$	(λ, k)
	(4)	1	$((4), 1)$
	$(3, 1)$	1	$((3, 1), 2)$
	$(2, 2)$	2	$((2, 2), 1), ((2, 2), 2)$
	$(2, 1, 1)$	2	$((2, 1, 1), 2), ((2, 1, 1), 3)$
	$(1, 1, 1, 1)$	4	$((1, 1, 1, 1), 1),\quad ((1, 1, 1, 1), 2)$ $((1, 1, 1, 1), 3),\quad ((1, 1, 1, 1), 4)$
Total	5	10	10

From the definition, it is easy to see that the number of marked partitions of n is equal to $spt(n)$.

Problem

The next problem is that

Problem

How to divide the set of marked partitions of $5n+4$ (*or* $7n+5$) *into five* (*or seven*) *equinumerous classes?*

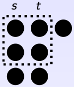
Here we first introduce a combinatorial structure called doubly marked partition. We then define $N_S(m,n)$ on doubly marked partitions and build a connection between marked partitions and doubly marked partitions. We then divide these marked partitions according to the spt-crank of doubly marked partitions.

William Y.C. Chen, Kathy Q. Ji, and Wenston J.T. Zang, The spt-crank for ordinary partitions, J. Reine Angew. Math. 711 (2016) 231–249.

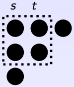
We find that $N_S(m,n)$ can be interpreted in terms of the doubly marked partitions of n with spt-crank m.

Definition. A doubly marked partition of n is a triplet (λ, s, t), where

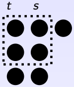
- λ is an ordinary partition of n;
- $1 \le s \le d(\lambda)$;
- $s \le t \le \lambda_1$;
- $\lambda'_s = \lambda'_t$.



((3, 2, 2), 1, 2)



((3, 2, 1), 1, 2)



((3, 2, 2), 2, 1)

The definition of spt-crank

We give the definition of spt-crank in terms of doubly marked partitions.

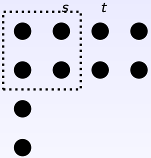
Definition (Chen, Ji and Zang, 2016)

Let (μ, s, t) be a doubly marked partition. Then the spt-crank of (μ, s, t), denoted $d(\mu, s, t)$, is defined by

$$d(\mu, s, t) = \mu'_s - \mu_{\mu'_s - s + 1} + t - 2s + 1.$$

Let $((4, 4, 1, 1), 2, 3)$ be a doubly marked partition of 10, its spt-crank is equal to

$$\begin{aligned} & d((4, 4, 1, 1), 2, 3) \\ & = \mu'_2 - \mu_{\mu'_2 - 2 + 1} + 3 - 4 + 1 \\ & = 2 - 4 + 3 - 4 + 1 \\ & = -2. \end{aligned}$$



Example

S-partition	weight	crank	doubly marked partition	crank
$((1),(1,1,1),\emptyset)$	$+1$	3	$((1,1,1,1),1,1)$	3
$((1),(2,1),\emptyset)$	$+1$	2	$((2,1,1),1,1)$	2
$((1),(1,1),(1))$	$+1$	1	$((3,1),1,1)$	1
$((1),(3),\emptyset)$	$+1$	1	$((2,2),1,2)$	1
$((2,1),(1),\emptyset)$	-1	1		
$((2),(2),\emptyset)$	$+1$	1		
$((1),(2),(1))$	$+1$	0	$((2,2),1,1)$	0
$((1),(1),(2))$	$+1$	0	$((4),1,4)$	0
$((3,1),\emptyset,\emptyset)$	-1	0		
$((4),\emptyset,\emptyset)$	$+1$	0		

Example

S-partition	weight	crank	doubly marked partition	crank
$((1),(1),(1,1))$	$+1$	-1	$((2,2),2,2)$	-1
$((1),\emptyset,(3))$	$+1$	-1	$((4),1,3)$	-1
$((2,1),\emptyset,(1))$	-1	-1		
$((2),\emptyset,(2))$	$+1$	-1		
$((1),\emptyset,(2,1))$	$+1$	-2	$((4),1,2)$	-2
$((1),\emptyset,(1,1,1))$	$+1$	-3	$((4),1,1)$	-3

Connection

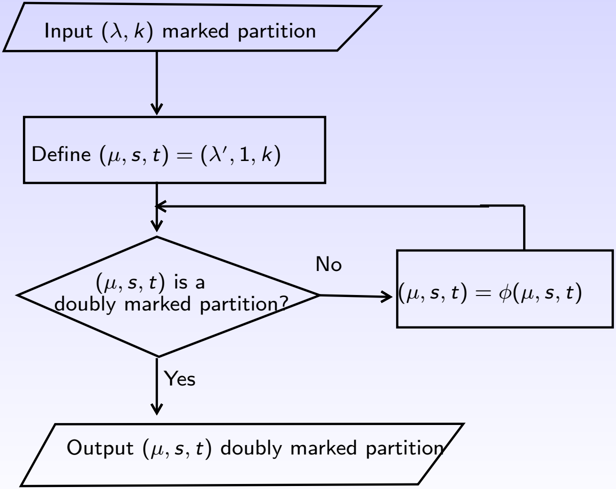
We have the following result.

Theorem (Chen, Ji and Zang, 2016)

There is a bijection $\triangle$ between the set of marked partitions of n and the set of doubly marked partitions of n.

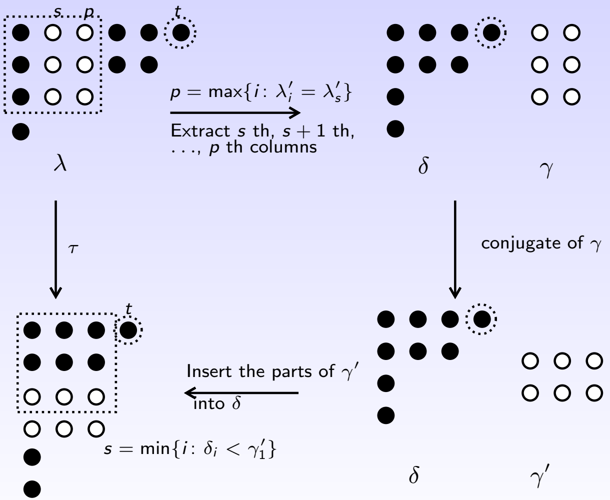
The bijection $\triangle$

The bijection $\triangle$ is constructed as shown in the following figure. Let (λ, k) be a marked partition and $(\mu, a, b) = \triangle(\lambda, k)$.



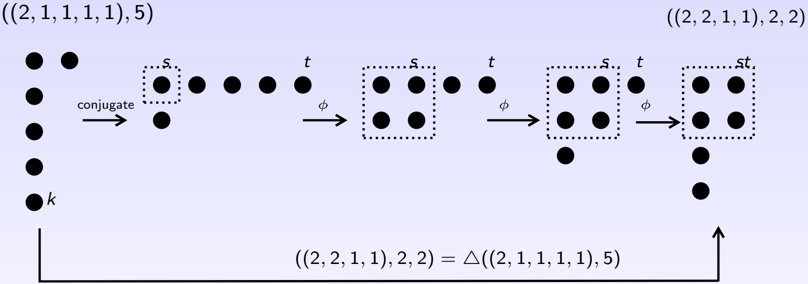
The map ϕ

The map ϕ: $((6,5,3,1),2,6) \to ((4,3,3,3,1,1),3,4)$.



An example of the bijection $\triangle$

We give an example to explain how to get a double marked partition $((2,2,1,1),2,2)$ from a marked partition $((2,1,1,1,1),5)$.



We have shown that this process is well-defined and reversible.

Example-1

For example, for $n = 4$, we list ten marked partitions of 4, the corresponding doubly marked partitions, and the spt-crank modulo 5.

(λ, k)	$(\mu, s, t) = \triangle(\lambda, k)$	$d(\mu, s, t)$	$d(\mu, s, t) \bmod 5$
$((1,1,1,1),4)$	$((4),1,4)$	0	0
$((2,2),1)$	$((2,2),1,1)$	0	0
$((3,1),2)$	$((3,1),1,1)$	1	1
$((2,2),2)$	$((2,2),1,2)$	1	1
$((1,1,1,1),1)$	$((4),1,1)$	-3	2
$((2,1,1),2)$	$((2,1,1),1,1)$	2	2
$((1,1,1,1),2)$	$((4),1,2)$	-2	3
$((4),1)$	$((1,1,1,1),1,1)$	3	3
$((2,1,1),3)$	$((2,2),2,2)$	-1	4
$((1,1,1,1),3)$	$((4),1,3)$	-1	4

Example-2

For another example, for $n = 5$, we list fourteen marked partitions of 5, the corresponding doubly marked partitions, and the spt-crank modulo 7.

(λ, k)	$(\mu, s, t) = \triangle(\lambda, k)$	$d(\mu, s, t)$	$d(\mu, s, t) \mod 7$
$((3,1,1),2)$	$((3,2),1,1)$	0	0
$((1,1,1,1,1),5)$	$((5),1,5)$	0	0
$((3,1,1),3)$	$((3,2),1,2)$	1	1
$((4,1),2)$	$((4,1),1,1)$	1	1
$((3,2),2)$	$((3,1,1),1,1)$	2	2
$((2,2,1),3)$	$((2,2,1),1,1)$	2	2
$((2,1,1,1),2)$	$((2,1,1,1),1,1)$	3	3
$((1,1,1,1,1),1)$	$((5),1,1)$	-4	3
$((5),1)$	$((1,1,1,1,1),1,1)$	4	4
$((1,1,1,1,1),2)$	$((5),1,2)$	-3	4

Example-2

(λ, k)	$(\mu, a, b) = \triangle(\lambda, k)$	$d(\mu, a, b)$	$d(\mu, a, b) \mod 7$
$((2,1,1,1),3)$	$((3,2),2,2)$	-2	5
$((1,1,1,1,1),3)$	$((5),1,3)$	-2	5
$((1,1,1,1,1),4)$	$((5),1,4)$	-1	6
$((2,1,1,1),4)$	$((2,2,1),2,2)$	-1	6

The conjecture on spt-crank

In 2013, Andrews, Dyson and Rhoades raised the following conjecture on $N_S(m,n)$:

Conjecture (Andrews, Dyson and Rhoades, 2013)

For $m \geq 0$ and $n \geq 1$:

$$N_S(m,n) \geq N_S(m+1,n).$$

They proved this conjecture for sufficiently large n:

Theorem (Andrews, Dyson and Rhoades, 2013)

For fixed $m \geq 0$,

$$N_S(m,n) - N_S(m+1,n) \sim \frac{(2m+1)\pi^2}{384\sqrt{3}n^2} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \quad \text{as } n \to \infty.$$

They also showed that this conjecture is equivalent to the following inequality:

Conjecture (Andrews, Dyson and Rhoades, 2013)

For $m \geq 0$ and $n \geq 1$, we have

$$M(\leq m, n) \leq N(\leq m, n),$$

where $N(\leq m, n)$ denote the number of partitions of n with rank less than or equal to m, and $M(\leq m, n)$ denote the number of partitions of n with crank less than or equal to m.

When $m = 0$, this inequality was conjectured by Kaavya.

G.E. Andrews, F.J. Dyson, R.C. Rhoades, On the distribution of the spt-crank, Mathematics 1 (2013) 76–98.

S.J. Kaavya, Crank 0 partitions and the parity of the partition function, Int. J. Number Theory 7 (2011) 793–801.

Confirmation of the conjecture

We build an injection from the set of partitions of n with crank less than or equal to m to the set of partitions of n with rank less than or equal to m. Thus we give a combinatorial proof of the following theorem.

Theorem (Chen, Ji and Zang, 2015)

For all $m \geq 0$ and $n \geq 1$, we have

$$M(\leq m, n) \leq N(\leq m, n).$$

William Y.C. Chen, Kathy Q. Ji and Wenston J.T. Zang, Proof of the Andrews-Dyson-Rhoades conjecture on the spt-crank, Adv. Math. 270(2015) 60–96.

The inequality

Actually, the inequality of rank and crank mentioned above has already been conjectured in 2009, by Bringmann and Mahlburg, as follows:

Conjecture (Bringmann and Mahlburg)

For all $n \geq 1$ and $m \geq 0$, we have

$$N(\leq m-1, n) \leq M(\leq m, n) \leq N(\leq m, n).$$

We give a combinatorial proof of the first half of the inequality. Thus the conjecture has been confirmed.

K. Bringmann, K. Mahlburg, Inequalities between ranks and cranks, Proc. Amer. Math. Soc. 137 (2009), no. 8, 2567–2574.

W.Y.C. Chen, K.Q. Ji and W.J.T. Zang, Nearly equal distributions of the rank and the crank of partitions, to appear.

The property of τ_n

This inequality implies a bijection $\tau_n : P(n) \to P(n)$, which has the following interesting property:

Theorem (Bringmann and Mahlburg, 2009)

For any $|\lambda| = n$, we have:

$$|crank(\lambda)| - |rank(\tau_n(\lambda))| = 0 \text{ or } 1.$$

Bringmann and Mahlburg pointed that the above property of τ_n leads to an upper bound of spt-function:

Theorem (Bringmann and Mahlburg, 2009)

$$spt(n) \leq \sqrt{2n}p(n).$$

More congruences

Garvan found the following congruences of the spt-function

Theorem (Garvan, 2010)

For $n \geq 0$,

$$\begin{aligned}
\mathrm{spt}(11 \cdot 19^4 \cdot n + 22006) &\equiv 0 \pmod{11},\\
\mathrm{spt}(17 \cdot 7^4 \cdot n + 243) &\equiv 0 \pmod{17},\\
\mathrm{spt}(19 \cdot 5^4 \cdot n + 99) &\equiv 0 \pmod{19},\\
\mathrm{spt}(29 \cdot 13^4 \cdot n + 18583) &\equiv 0 \pmod{29},\\
\mathrm{spt}(31 \cdot 29^4 \cdot n + 409532) &\equiv 0 \pmod{31},\\
\mathrm{spt}(37 \cdot 5^4 \cdot n + 1349) &\equiv 0 \pmod{37}.
\end{aligned}$$

F.G. Garvan, Congruences for Andrews' smallest parts partition function and new congruences for Dyson's rank, Int. J. Number Theory 6 (2) (2010) 281–309.

General case

Bringmann proved the following assertion.

Theorem (Bringmann, 2008)

For any prime $\ell \geq 5$, there are infinitely many arithmetic progressions $an + b$ such that

$$\mathrm{spt}(an+b) \equiv 0 \pmod{\ell}.$$

This assertion is reminiscent to the following result obtained by Ono.

Theorem (Ono, 2000)

For any prime $\ell \geq 5$, there are infinitely many arithmetic progressions $an + b$ such that

$$p(an+b) \equiv 0 \pmod{\ell}.$$

K. Bringmann, On the explicit construction of higher deformations of partition statistics, Duke Math. J. 144 (2) (2008) 195–233.

K. Ono, Distribution of the partition function modulo m, Ann. of Math. (2) 151 (1) (2000) 293–307.

Explicit Ramanujan-type congruence

Ono derived the following Ramanujan-type congruences of $\mathrm{spt}(n)$ modulo ℓ for any prime $\ell \geq 5$.

Theorem (Ono, 2011)

Let $\ell \geq 5$ be a prime and let $\left(\frac{\bullet}{\circ}\right)$ denote the Legendre symbol.

(i) *For $n \geq 1$, if $\left(\frac{-n}{\ell}\right) = 1$,*

$$\mathrm{spt}\left((\ell^2 n + 1)/24\right) \equiv 0 \pmod{\ell}.$$

(ii) *For $n \geq 0$,*

$$\mathrm{spt}\left((\ell^3 n + 1)/24\right) \equiv \left(\frac{3}{\ell}\right) \mathrm{spt}\left((\ell n + 1)/24\right) \pmod{\ell}.$$

K. Ono, Congruences for the Andrews spt-function, Proc. Natl. Acad. Sci. USA 108 (2) (2011) 473–476.

Explicit Ramanujan-type congruence on prime powers

Ahlgren, Bringmann and Lovejoy extended the above result to any prime power.

Theorem (Ahlgren, Bringmann and Lovejoy, 2011)

Let $\ell \geq 5$ be a prime and let $m \geq 1$.

(i) *For $n \geq 1$, if $\left(\frac{-n}{\ell}\right) = 1$,*

$$\mathrm{spt}\left((\ell^{2m} n + 1)/24\right) \equiv 0 \pmod{\ell^m}.$$

(ii) *For $n \geq 0$,*

$$\mathrm{spt}\left((\ell^{2m+1} n + 1)/24\right) \equiv \left(\frac{3}{\ell}\right) \mathrm{spt}\left((\ell^{2m-1} n + 1)/24\right) \pmod{\ell^m}.$$

S. Ahlgren, K. Bringmann and J. Lovejoy, ℓ-adic properties of smallest parts functions, Adv. Math. 228 (1) (2011) 629–645.

The congruences of powers of $5, 7$ and 13.

Garvan gave the following congruences of powers of $5, 7$ and 13.

Theorem (Garvan, 2012)

Let a, b, c be positive integers, and let δ_a, λ_b and γ_c be the least nonnegative residues of the reciprocals of 24 mod 5^a, 7^b and 13^c respectively. Then

$$\mathrm{spt}(5^a n + \delta_a) \equiv 0 \pmod{5^{\lfloor \frac{a+1}{2} \rfloor}},$$

$$\mathrm{spt}(7^b n + \lambda_b) \equiv 0 \pmod{7^{\lfloor \frac{b+1}{2} \rfloor}},$$

$$\mathrm{spt}(13^c n + \gamma_c) \equiv 0 \pmod{13^{\lfloor \frac{c+1}{2} \rfloor}}.$$

F.G. Garvan, Congruences for Andrews' spt-function modulo powers of 5, 7 and 13, Trans. Amer. Math. Soc. 364 (9) (2012) 4847–4873.

The parity of the spt-function

For $n \geq 1$, the parity of $\mathrm{spt}(n)$ is determined by Folsom and Ono. They obtained the following characterization of the parity of $\mathrm{spt}(n)$.

Theorem (Folsom and Ono, 2008)

The function $\mathrm{spt}(n)$ is odd if and only if $24n - 1 = pm^2$, where m is an integer and $p \equiv 23 \pmod{24}$ is prime.

As pointed out by Andrews, Garvan and Liang, the above theorem contains an error.

Example

For $n = 507$, it is clear that $507 \times 24 - 1 = 12167 = 23 \times 23^2 = pm^2$, where $p = m = 23$. Obviously, 507 satisfies the condition of the above theorem. But $\mathrm{spt}(507) = 6047032773755628522 5064$ is even.

A. Folsom and K. Ono, The spt-function of Andrews, Proc. Natl. Acad. Sci. USA 105 (51) (2008) 20152–20156.

The correct parity of the spt-function

Andrews, Garvan and Liang corrected Folosom and Ono's result as given below.

Theorem (Andrews, Garvan and Liang, 2013)

The function $\mathrm{spt}(n)$ *is odd if and only if* $24n-1=p^{4a+1}m^2$ *for some prime* $p \equiv 23 \pmod{24}$ *and some integers* a, m *with* $(p, m) = 1$.

G.E. Andrews, F.G. Garvan and J. Liang, Self-conjugate vector partitions and the parity of the spt-function, Acta Arith. 158 (3) (2013) 199–218.

The congruence of modulo 3

On the congruences of modulo 3, Folsom and Ono gave the following result:

Theorem (Folsom and Ono, 2008)

Let $\ell \geq 5$ be a prime such that $\ell \equiv 2 \pmod 3$. If $0 < k < \ell - 1$, then for $n \geq 0$,

$$\mathrm{spt}(\ell^4 n + \ell^3 k - (\ell^4 - 1)/24) \equiv 0 \pmod 3.$$

For example, for $\ell = 5$, we may deduce the following congruences:

$$\begin{aligned} \mathrm{spt}(625n + 99) &\equiv \mathrm{spt}(625n + 224) \equiv \mathrm{spt}(625n + 349) \\ &\equiv \mathrm{spt}(625n + 474) \equiv 0 \pmod 3. \end{aligned}$$

The variations of the spt-function

The k-th symmetrized rank function $\eta_k(n)$ was defined by Andrews as

$$\eta_k(n) = \sum_{m=-n}^{n} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N(m,n).$$

Garvan introduced the k-th symmetrized crank function $\mu_k(n)$ as follows:

$$\mu_k(n) = \sum_{m=-n}^{n} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M(m,n).$$

G.E. Andrews, Partitions, Durfee symbols, and the Atkin-Garvan moments of ranks, Invent. Math. 169 (1) (2007) 37–73.

F.G. Garvan, Higher order spt-functions, Adv. Math. 228 (1) (2011) 241–265.

The higher order spt-function

Garvan introduced the higher order spt-function $\mathrm{spt}_k(n)$ in terms of $\mu_k(n)$ and $\eta_k(n)$.

Definition (Garvan, 2011)

For $k \geq 1$, define

$$\mathrm{spt}_k(n) = \mu_{2k}(n) - \eta_{2k}(n).$$

Garvan proved that

$$\mu_{2k}(n) \geq \eta_{2k}(n),$$

which implies that $\mathrm{spt}_k(n) \geq 0$.

The congruences of the higher spt-function

Garvan obtained congruences of $\mathrm{spt}_2(n)$, $\mathrm{spt}_3(n)$ and $\mathrm{spt}_4(n)$.

Theorem (Garvan, 2011)

For $n \geq 1$,

$$\begin{aligned}
\mathrm{spt}_2(n) &\equiv 0 \pmod 5, && \text{if } n \equiv 0,1,4 \pmod 5,\\
\mathrm{spt}_2(n) &\equiv 0 \pmod 7, && \text{if } n \equiv 0,1,5 \pmod 7,\\
\mathrm{spt}_2(n) &\equiv 0 \pmod{11}, && \text{if } n \equiv 0 \pmod{11},\\
\mathrm{spt}_3(n) &\equiv 0 \pmod 7, && \text{if } n \not\equiv 3,6 \pmod 7,\\
\mathrm{spt}_3(n) &\equiv 0 \pmod 2, && \text{if } n \equiv 1 \pmod 4,\\
\mathrm{spt}_4(n) &\equiv 0 \pmod 3, && \text{if } n \equiv 0 \pmod 3.
\end{aligned}$$

Garvan also provided a combinatorial interpretation of $\mathrm{spt}_k(n)$.

The ospt-function

Andrews, Chan and Kim introduced the modified rank and crank moments $N_j^+(n)$ and $M_j^+(n)$ by considering the following unilateral sums:

$$N_j^+(n) = \sum_{m\geq 0} m^j N(m,n)$$

and

$$M_j^+(n) = \sum_{m\geq 0} m^j M(m,n).$$

They defined the ospt-function $\mathrm{ospt}(n)$ as given below:

Definition (Andrews, Chan and Kim, 2013)

For $n \geq 1$,

$$\mathrm{ospt}(n) = M_1^+(n) - N_1^+(n).$$

G.E. Andrews, S.H. Chan and B. Kim, The odd moments of ranks and cranks, J. Combin. Theory Ser. A 120 (1) (2013) 77–91.

The positivity of the ospt-function

Andrews, Chan and Kim obtained the following inequality.

Theorem (Andrews, Chan and Kim, 2013)

For all positive integer n and j,

$$M_j^+(n) > N_j^+(n).$$

This yields that $\mathrm{ospt}(n) > 0$. They also gave the combinatorial interpretation of $\mathrm{ospt}(n)$ in terms of even and odd strings of a partition. Using this combinatorial interpretation, Bringmann and Mahlburg proved a monotone property of $\mathrm{ospt}(n)$ as given below.

Theorem (Bringmann and Mahlburg, 2014)

For $n \geq 1$,

$$\mathrm{ospt}(n+1) \geq \mathrm{ospt}(n).$$

K. Bringmann and K. Mahlburg, Asymptotic inequalities for positive crank and rank moments, Trans. Amer. Math. Soc. 366 (2) (2014) 1073–1094.

The inequalities of the ospt-function

Chan and Mao proved the following inequalities of $\mathrm{ospt}(n)$.

Theorem (Chan and Mao, 2014)

We have

$$\mathrm{ospt}(n) > \frac{p(n)}{4} + \frac{N(0,n)}{2} - \frac{M(0,n)}{4} \quad \textit{for } n \geq 8,$$

$$\mathrm{ospt}(n) < \frac{p(n)}{4} + \frac{N(0,n)}{2} - \frac{M(0,n)}{4} + \frac{N(1,n)}{2} \quad \textit{for } n \geq 7,$$

$$\mathrm{ospt}(n) < \frac{p(n)}{2} \quad \textit{for } n \geq 3.$$

S.H. Chan and R. Mao, Inequalities for ranks of partitions and the first moment of ranks and cranks of partitions, Adv. Math. 258 (2014) 414–437.

The asymptotic properties of the spt-function

By applying the circle method to the second symmetrized rank moment $\eta_2(n)$, Bringmann obtained an asymptotic expression of the spt-function $\mathrm{spt}(n)$.

Theorem (Bringmann, 2008)

As $n \to \infty$,

$$\mathrm{spt}(n) \sim \frac{\sqrt{6}}{\pi}\sqrt{n}p(n) \sim \frac{1}{2\sqrt{2}\pi\sqrt{n}} e^{\pi\sqrt{\frac{2n}{3}}}.$$

K. Bringmann, On the explicit construction of higher deformations of partition statistics, Duke Math. J. 144 (2) (2008) 195–233.

The exact relation of the spt-function

Ahlgren and Andresen obtained an exact expression for the spt-function.

Theorem (Ahlgren and Andresen, 2016)

For $n \geq 1$,

$$\mathrm{spt}(n) = \frac{\pi}{6}(24n-1)^{\frac{1}{4}} \sum_{c=1}^{\infty} \frac{A_c(n)}{c} (I_{1/2} - I_{3/2}) \left(\frac{\pi\sqrt{24n-1}}{6c} \right),$$

where I_ν is the I-Bessel function, $A_c(n)$ is the Kloosterman sum

$$A_c(n) = \sum_{\substack{d \mod c \\ (d,c)=1}} e^{\pi i s(d,c) - 2i\pi \frac{dn}{c}}.$$

S. Ahlgren and N. Andersen, Algebraic and transcendental formulas for the smallest parts function, Adv. Math. 289 (2016) 411–437.

The asymptotic property of $\mathrm{spt}_k(n)$

The asymptotic property of $\mathrm{spt}_k(n)$ was first conjectured by Bringmann and Mahlburg, and then confirmed by Bringmann, Mahlburg and Rhoades.

Theorem (Bringmann, Mahlburg and Rhoades, 2011)

As $n \to \infty$,

$$\mathrm{spt}_k(n) \sim \beta_{2k} n^{k-\frac{1}{2}} p(n),$$

where $\beta_{2k} \in \frac{\sqrt{6}}{\pi}\mathbb{Q}$ is positive.

K. Bringmann and K. Mahlburg, Inequalities between ranks and cranks, Proc. Amer. Math. Soc. 137 (8) (2009) 2567–2574.

K. Bringmann, K. Mahlburg and R.C. Rhoades, Asymptotics for rank and crank moments, Bull. London Math. Soc. 43 (4) (2011) 661–672.

The asymptotic property of $\mathrm{ospt}(n)$

Bringmann and Mahlburg derived the asymptotic formula of $\mathrm{ospt}(n)$ as given below.

Theorem (Bringmann and Mahlburg, 2014)

As $n \to \infty$,

$$\mathrm{ospt}(n) \sim \frac{p(n)}{4} \sim \frac{1}{16\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}.$$

K. Bringmann and K. Mahlburg, Asymptotic inequalities for positive crank and rank moments, Trans. Amer. Math. Soc. 366 (2) (2014) 1073–1094.

The log-concavity of $p(n)$

DeSalvo and Pak proved that the partition function $p(n)$ satisfies the log-concave property for $n \geq 26$.

Theorem (DeSalvo and Pak, 2015)

For $n \geq 26$,

$$p(n)^2 > p(n-1)p(n+1).$$

They also proved the following theorem.

Theorem (DeSalvo and Pak, 2015)

For $n \geq 2$,

$$\frac{p(n-1)}{p(n)}\left(1+\frac{1}{n}\right) > \frac{p(n)}{p(n+1)}.$$

S. DeSalvo and I. Pak, Log-concavity of the partition function, Ramanujan J. 38 (1) (2015) 61–73.

Results on $p(n)$

DeSalvo and Pak further proved that the term $(1 + 1/n)$ in the above theorem can be improved to $(1 + O(n^{-3/2}))$.

Theorem (DeSalvo and Pak, 2015)

For $n > 6$,

$$\frac{p(n-1)}{p(n)}\left(1 + \frac{240}{(24n)^{3/2}}\right) > \frac{p(n)}{p(n+1)}.$$

DeSalvo and Pak conjectured that the coefficient of $1/n^{3/2}$ can be improved to $\pi/\sqrt{24}$, which was proved by Chen, Wang and Xie.

Theorem (Chen, Wang and Xie, 2016)

For $n \geq 45$,

$$\frac{p(n-1)}{p(n)}\left(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{p(n)}{p(n+1)}.$$

W.Y.C. Chen, L.X.W. Wang and G.Y.B. Xie, Finite differences of the logarithm of the partition function, Math. Comp. 85 (298) (2016) 825–847.

Similar inequalities on $\mathrm{spt}(n)$

We now present some conjectures on $\mathrm{spt}(n)$.

Conjecture

(1) *For* $n \geq 36$,

$$\mathrm{spt}(n)^2 > \mathrm{spt}(n-1)\,\mathrm{spt}(n+1).$$

(2) *For* $n \geq 13$,

$$\frac{\mathrm{spt}(n-1)}{\mathrm{spt}(n)}\left(1+\frac{1}{n}\right) > \frac{\mathrm{spt}(n)}{\mathrm{spt}(n+1)}.$$

(3) *For* $n \geq 73$,

$$\frac{\mathrm{spt}(n-1)}{\mathrm{spt}(n)}\left(1+\frac{\pi}{\sqrt{24}n^{3/2}}\right) > \frac{\mathrm{spt}(n)}{\mathrm{spt}(n+1)}.$$

(4) *For* $n > m > 1$,

$$\mathrm{spt}(n)^2 > \mathrm{spt}(n-m)\,\mathrm{spt}(n+m).$$

(5) *If* a, b *are integers with* $a, b > 1$ *and* $(a, b) \neq (2, 2)$ *or* $(3, 3)$, *then*

$$\mathrm{spt}(a)\,\mathrm{spt}(b) > \mathrm{spt}(a+b).$$

Confirmation of the conjecture

Recently Madeline Locus gives the following upper-bound and lower-bound on $\mathrm{spt}(n)$.

Theorem (Madeline Locus)

For each $a \in \mathbb{Z}^+$ and $k \in \mathbb{Z}^+$, there is a $B_k(a)$ such that for all $n \geq B_k(a)$,

$$\frac{\sqrt{3}}{\pi\sqrt{24n-1}}\left(1-\frac{1}{an^k}\right)e^{\mu(n)} < \mathrm{spt}(n) < \frac{\sqrt{3}}{\pi\sqrt{24n-1}}\left(1+\frac{1}{an^k}\right)e^{\mu(n)},$$

where $\mu(n) = \frac{\pi}{6}\sqrt{24n-1}$.

With the aid of the above estimate, Locus announces a proof of all the above conjectures on spt-function.

M. Locus, Inequalities satisfied by the Andrews spt-function, preprint.

Higher order Turán inequality

Recall that a sequence $\{a_n\}_{n\geq 0}$ satisfies the higher order Turán inequality if for $n \geq 1$,

$$4(a_n^2 - a_{n-1}a_{n+1})(a_{n+1}^2 - a_n a_{n+2}) - (a_n a_{n+1} - a_{n-1}a_{n+2})^2 > 0.$$

Numerical evidence indicates that both $p(n)$ and $\mathrm{spt}(n)$ satisfy the higher order Turán inequality.

Conjecture

For $n \geq 95$, $p(n)$ satisfies the higher order Turán inequality, whereas $\mathrm{spt}(n)$ satisfies the higher order Turán inequality for $n \geq 108$.

THANK YOU!